P-way Determinants, with an Application to Transvectants.

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In this paper an extended definition of a determinant is given which applies to determinants of more than three dimensions, and enables us to remove the restriction in Cayley's law of multiplication and to set up a new case in Scott's law of multiplication. New formulas are obtained for the known process of decomposition of a determinant into determinants of fewer dimensions, and a new process called crossed decomposition is described. Fresh light is thrown upon the function known as a "determinant-permanent," a limitation hitherto thought necessary being done away. Finally a generalization to p dimensions is made of Metzler's theorem in two dimensions concerning a determinant, each of whose elements is the product of k factors.

We lead up to these matters by a brief statement of the elementary theory of 3-way or cubic determinants and permanents.

I. THREE-WAY DETERMINANTS AND PERMANENTS.

1. *Definitions and Fundamental Properties.*—Elements, n^3 in number, can be arranged in a 3-way matrix of order n having n^2 *rows*, n^2 *columns*, and n^2 *normals*; all are called *files*. The matrix divides up into n *strata*, or layers that contain rows and columns and are pierced by normals; and into n *row-normal layers* pierced by columns; and into n *column-normal layers* pierced by rows; all are called *layers*. In triple-index notation, a_{ijk} denotes the element in the i-th stratum, the j-th row-normal layer, and the k-th column-normal layer. To represent 3-way matrices of successive orders, we write:

$$\left\| \begin{array}{cc|cc} a_{111} & a_{112} & a_{211} & a_{212} \\ a_{121} & a_{122} & a_{221} & a_{222} \end{array} \right\|, \quad \left\| \begin{array}{ccc|ccc|ccc} a_{111} & a_{112} & a_{113} & a_{211} & a_{212} & a_{213} & a_{311} & a_{312} & a_{313} \\ a_{121} & a_{122} & a_{123} & a_{221} & a_{222} & a_{223} & a_{321} & a_{322} & a_{323} \\ a_{131} & a_{132} & a_{133} & a_{231} & a_{232} & a_{233} & a_{331} & a_{332} & a_{333} \end{array} \right\|, \text{ etc.}$$

Two or more elements are *conjunctive* if no two of them lie in the same layer of any direction; n conjunctive elements are *perjunctive* and form a *transversal.* When we speak of a determinant of a matrix and use the word "transversal," we mean the product of these elements. The *locant* of an

element is the set of indices which locate it in the matrix. If the locants of a perjunctive set of elements be written in a column, the three subcolumns, of n indices each, are called *ranges*, and the whole is called the locant of the set (transversal); *e. g.*, $a_{122}\, a_{313}\, a_{231}$, ranges 132, 213, 231. The *sign of a range* is $+$ or $-$ according as there is an even or an odd number of inversions of order in the range.

The *determinant* of a 3-way matrix is the algebraic sum of its transversals, each having the sign which is the product of the signs of its second and third ranges.

The *permanent* of a 3-way matrix is the sum of its transversals.

The determinant and permanent being homogeneous linear functions of the elements of any layer, we have obvious theorems as to factors of layers, and as to separation into a sum of 3-way determinants or permanents when there are polynomial elements, and, conversely, as to addition of determinants and of permanents. An interchange of strata in a determinant, or of any parallel layers in a permanent, does not change its value; but an interchange of any two parallel layers other than strata, in a determinant, changes its sign. Hence, if two such layers are alike the determinant vanishes. Hence a multiple of such a layer may be added to any parallel layer without changing the value of the determinant.

A *minor* is formed by striking out an equal number of layers of each direction. It is obvious what we mean by *conjunctive minors* and by *perjunctive minors*. A perjunctive set of minors, formed into a product, with the proper sign, is equal to the sum of a certain number of terms of the determinant; the sign is the sign of that term of the determinant whose elements are the elements in the main diagonals of the minors. In the simple case of an element a_{ijk} and its complementary minor, the sign is $(-1)^{j+k}$. In consequence, the Laplacean expansion of a determinant is formed by partitioning the matrix into two or more sets of parallel layers and forming all possible perjunctive sets of minors occupying the sets of layers, one minor in each set.

2. *Decomposition.*—(i) A 3-way determinant Δ of order n can be decomposed into the sum of $n!$ 2-way determinants whose rows are rows of Δ. For, arranging n perjunctive rows of Δ in the order of the row-normal layers in which they lie, we see that the 2-way determinant

$$\begin{vmatrix} a_{i'11} & a_{i'12} & \dots & a_{i'1n} \\ a_{i''21} & a_{i''22} & \dots & a_{i''2n} \\ \dots & \dots & \dots & \dots \\ a_{i^{(n)}n1} & a_{i^{(n)}n2} & \dots & a_{i^{(n)}nn} \end{vmatrix},$$

whose matrix they form consists of $n!$ terms of Δ, and that the totality of such 2-way determinants consists of all the terms of Δ. They are called *components of* Δ.

(ii) If the rows are arranged in the order of the *strata* in which they lie, then each determinant must have prefixed the sign of the j-range (denoted by $\pm_j$) in the locant of the set of rows.

(iii) (iv) There will be two corresponding forms of decomposition into 2-way determinants whose columns are columns of Δ.

(v) We can also decompose Δ into an algebraic sum of 2-way permanents. Arrange n perjunctive normals in any order, and to the permanent whose matrix they form, prefix the sign of their locant; for, clearly, all of those $n!$ terms of Δ which lie in a perjunctive set of normals have the same sign.

$$\text{(ii)}\ \pm_j \begin{vmatrix} a_{1j'1} & a_{1j'2} & \cdots \\ a_{2j''1} & a_{2j''2} & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix}; \quad \text{(iii)} \begin{vmatrix} a_{i'11} & a_{i''12} & \cdots \\ a_{i'21} & a_{i''22} & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix};$$

$$\text{(iv)}\ \pm_k \begin{vmatrix} a_{11k'} & a_{21k''} & \cdots \\ a_{12k'} & a_{22k''} & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix}; \quad \text{(v)}\ \pm_j \pm_k \overset{+}{\left|\begin{matrix} a_{1j'k'} & a_{2j'k'} & \cdots \\ a_{1j''k''} & a_{2j''k''} & \cdots \\ \cdots & \cdots & \cdots \end{matrix}\right|}^{+*}.$$

A 3-way *permanent* may of course be decomposed by rows, columns, or normals, into a sum of 2-way permanents.

3. *Element-Multiplication.*—(i) (Scott's† law of multiplication). The product of two 2-way determinants (or permanents) A and B of order n is expressible as a 3-way determinant (or permanent) C of order n wherein

$$c_{ijk} \equiv a_{ij} b_{ik}.$$

EXAMPLE: $$\overset{+}{\left|\begin{matrix} a_{11}a_{12} \\ a_{21}a_{22} \end{matrix}\right|} \cdot \overset{+}{\left|\begin{matrix} b_{11}b_{12} \\ b_{21}b_{22} \end{matrix}\right|} = \overset{+}{\left|\begin{matrix} a_{11}b_{11} & a_{11}b_{12} \\ a_{12}b_{11} & a_{12}b_{12} \end{matrix}\right.}\left|\begin{matrix} a_{21}b_{21} & a_{21}b_{22} \\ a_{22}b_{21} & a_{22}b_{22} \end{matrix}\right|.$$

From the *prescription* $c_{ijk} \equiv a_{ij} b_{ik}$, whatever be the value of n, we see (i) noting the index i, that columns of A and columns of B are found in normals of C, (ii) noting the index j, that rows of A are found in columns of C and that elements of B run as factors through columns of C; and (iii) noting the index k, that rows of B are found in rows of C, and that elements of A run as factors through rows of C.

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* “$\overset{+}{|}\ |$” means a determinant or permanent; “$\overset{+}{|}\ \overset{+}{|}$” means a permanent.

† R. F. Scott, “On Cubic Determinants,” etc., *Proc. London Math. Soc.*, Vol. XI (1879), p. 17, at p. 23, paragraph 7. In paragraphs 8 and 9, Scott extends the rule so as to give the product of two determinants of p and q dimensions, respectively, in the form of a determinant of $p+q-1$ dimensions. In § 7 of the present paper this rule is extended to determinants as defined in § 5.

This method of examining a prescription will often give at once a good idea of a matrix and may suggest the best way of dealing with it.

In the present case we use decomposition (ii) of the preceding section and find that each component equals B multiplied by a term of A, giving $AB=C$.

(ii) The product of a 2-way determinant A and a 2-way permanent P of order n is expressible as a 3-way determinant C wherein

$$c_{ijk} \equiv a_{jk} p_{ik};$$

that is, a normal of C is a column of P with an a-factor; a column of C is a column of A with a p-factor; and a row of C is a row of A and a row of P. Using decomposition (v), we find that each component equals P multiplied by a term of A, and $AP=C$.

4. *File-Multiplication.*—(Cayley's * law of multiplication.) The product of a 3-way determinant A and a 2-way determinant B of order n is expressible as a 3-way determinant C of order n, wherein

$$c_{ijk} \equiv \sum_{l=1}^{n} a_{ijl} b_{kl};$$

that is, the matrix of each stratum of C is precisely what would result from using the familiar process of multiplying together two 2-way determinants into a 2-way determinant, row into row; one of the determinants being always B, and the other being the determinant of the matrix of the stratum of A corresponding to that of C. In brief, we multiply B into the strata of A to form C.

Using decomposition (i), we find that each component $C_r = A_r B$, whence $C=AB$.

II. Determinants and Permanents of p Dimensions.

5. *Definitions and Fundamental Properties.*—A p-way matrix (or a matrix of *class* p) of order n is formed of n^p elements:

$$\| a_{a_1 a_2 \ldots . a_p} \|_n^{(p)}.$$

The matrix can be separated into n layers, in any one of the p directions; a layer is a $(p-1)$-way matrix of n^{p-1} elements. Common to two layers of

* A. Cayley, "On the Theory of Determinants," *Trans. Cambridge Phil. Soc.*, Vol. VIII (1843), p. 75; *Coll. Math. Papers*, Vol. I, p. 63. See §2 of that paper. Cayley extends the rule so as to give the product of two determinants of p and q dimensions, respectively, in the form of a determinant of $p+q-2$ dimensions, but states that the rule is inapplicable when both p and q are odd; a restriction ignored by Scott in paragraph 9 of the paper cited in the first note, and by many others (see "Abrégé de la théorie des déterminants à n dimensions," par Maurice Lecat; Gand, Hoste, 1911, pp. 55 *et seq.*). In §8 of the present paper this rule is extended to determinants as defined in §5, such determinants serving to remove the restriction.

different directions is a $(p-2)$-way *sublayer;* common to three, a $(p-3)$-way sublayer; and so on, until we have, common to $p-1$ layers of different directions, a *file* of n elements, piercing the n parallel layers of the remaining direction. Parallel to this file are $n^{p-1}-1$ other files, together with it containing all the elements of the matrix. Files of the last direction are *rows.*

We speak of *conjunctive* and *perjunctive* elements, files, etc., of *transversals,* and of *locants* and *ranges,* as we did in the special case of a 3-way matrix.

Hitherto a p-way determinant has been defined as the algebraic sum of the transversals of a p-way matrix, the sign of a transversal being determined by arranging its elements in such an order that the values of a fixed index shall read $1, 2, \ldots, n$, and then taking the product of the signs of all the other ranges. It has then been shown that for a matrix of even class the same determinant will result, whatever be the fixed index, but that for a matrix of odd class a different determinant, in general, will result from a different choice of the fixed index. It has also been shown that interchange of layers denoted by the fixed index in a determinant of odd class leaves the value of the determinant unchanged, while the interchange of two layers of any other direction in such a determinant, or the interchange of two layers of any direction in a determinant of even class, changes the sign of the determinant.

It follows that without making the supposition that the elements of a transversal are first arranged in any particular order, we may say that the sign is the product (i) of the signs of all the ranges, in a determinant of even class, and (ii) of the signs of all but one (a fixed one) of the ranges, in a determinant of odd class.

From this point of view, we now generalize the definition of a determinant. We shall call an index (or range, or direction, or file, or set of layers) *signant* or *nonsignant* according as we do or do not take the order therein into account in fixing the sign of a term. In a 2-way determinant both indices are signant; in a 2-way permanent, both indices are nonsignant. In a 3-way determinant, two indices are signant. Passing to matrices of more dimensions than three, we see that it is possible not only to have signant all the indices if the class is even, and all but one if the class is odd, but to have signant a less number in either case, provided only that there be an even number that are signant—two indices in a 4-way or 5-way matrix, two or four indices in a 6-way or 7-way matrix, and so on. We therefore lay down the following:

Definition of a Determinant.—A determinant of a p-way matrix is the sum of all the terms that can be formed by taking a set of perjunctive elements

as factors and prefixing the product of the signs of an even number of chosen ranges.

A permanent might be viewed as one extreme, where the even number is zero; but file-multiplication and dependent processes have no application to permanents, and so we prefer to mention them explicitly when a theorem is true with regard to them, and to understand that a determinant has at least two signant indices.

By a *full-sign* determinant we shall mean a determinant as heretofore defined, with all, or all but one, of the indices signant, according as it is of even or odd class.

If in any determinant two layers of a signant direction be interchanged, the sign of the determinant is changed. Hence, if two such layers are alike, the determinant vanishes. Hence a multiple of one such layer may be added to another without changing the value of the determinant.

Both a determinant and a permanent are of course homogeneous linear functions of the elements of any layer and have the properties resulting.

6. *Decomposition.*—A p-way n-layer determinant Δ can be decomposed into the algebraic sum of $n!$ $(p-1)$-way determinants or permanents, as the case may be. Each of these components has $n!$ components, and so on. Ultimately we arrive at the expression of Δ as the algebraic sum of $(n!)^{p-2}$ 2-way determinants or permanents.

Let the matrix $\|a_{\alpha\beta\ldots\ldots\kappa\lambda}\|_n^{(p)}$ of Δ be divided up into its $(p-2)$-way sublayers of directions 1 and 2. Denote the sublayer common to the r-th layer of direction 1, and the s-th layer of direction 2 by $a_{rs0\ldots\ldots0}$. Take n perjunctive sublayers $a_{\alpha'10\ldots\ldots0}, a_{\alpha''20\ldots\ldots0}, \ldots\ldots, a_{\alpha^{(n)}n0\ldots\ldots0}$ to form a $(p-1)$-way matrix,

$$\left\| \begin{matrix} a_{\alpha'10\ldots\ldots0} \\ a_{\alpha''20\ldots\ldots0} \\ \cdots\cdots \\ a_{\alpha^{(n)}n0\ldots\ldots0} \end{matrix} \right\|.$$

This is a component of the matrix of Δ; there are $n!$ such components, and all of the locant of an element in Δ except the first index, is the locant of that element in each of the $(n-1)!$ components in which it occurs.

Denoting by the superscripts $\frown$ and $\smile$ the signant and nonsignant indices, and inserting a colon to isolate the locant of a component, we have:

$$\left.\begin{aligned}
&(1)\quad |a_{\widehat{\alpha}\widehat{\beta}\ldots\ldots}|_n^{(p)} = \sum_\alpha \pm_\alpha |a_{\alpha:\breve{\beta}\ldots\ldots}|_n^{(p-1)}; \\
&(2)\quad |a_{\widehat{\alpha}\breve{\beta}\ldots\ldots}|_n^{(p)} = \sum_\alpha \pm_\alpha |a_{\alpha:\widehat{\beta}\ldots\ldots}|_n^{(p-1)}; \\
&(3)\quad |a_{\breve{\alpha}\widehat{\beta}\ldots\ldots}|_n^{(p)} = \sum_\alpha \quad |a_{\alpha:\widehat{\beta}\ldots\ldots}|_n^{(p-1)}; \\
&(4)\quad |a_{\breve{\alpha}\breve{\beta}\ldots\ldots}|_n^{(p)} = \sum_\alpha \quad |a_{\alpha:\breve{\beta}\ldots\ldots}|_n^{(p-1)}.
\end{aligned}\right\} \qquad (D_1)$$

Each index beyond β is signant or nonsignant on the right according as it is signant or nonsignant on the left. In (1) and (2), $\pm_{\alpha}$ is the sign of the α-range in any transversal when the β-range reads $12\ldots.n$.

Briefly, if α is signant (nonsignant) the components are signed (unsigned) and the signancy of β is reversed (continued).

We see that the components of a determinant may be determinants or may be permanents; but that the components of a permanent must be permanents.

In verifying the formulas we must recall the fact that there are an even number of signant indices. Consider a term of $|a_{\alpha:\breve{\beta}\ldots.}|_n^{(p-1)}$ in formula (1). Let its elements be arranged so that the values of β are in the order $12\ldots.n$; then its sign is the product of the signs of the signant ranges beyond β. Prefixing $\pm_{\alpha}$, we find that we now have the sign proper to this term in $|a_{\overset{\frown}{\alpha}\overset{\frown}{\beta}\ldots.}|_n^{(p)}$. And this sign is preserved when the elements are permuted, provided that we make β nonsignant, because there are an even number of signant ranges beyond β.

If the process be repeated to give the $(p-2)$-way components, we shall have $(n!)^2$ matrices of the form

$$\left\|\begin{matrix} a_{\alpha'\beta'10\ldots.0} \\ a_{\alpha''\beta''20\ldots.0} \\ \cdots\cdots\cdots \\ a_{\alpha^{(n)}\beta^{(n)}n0\ldots.0} \end{matrix}\right\|.$$

To condense the corresponding formulas, we use a double superfix $\overset{\smile}{\frown}$ or $\overset{\frown}{\smile}$, the upper signs to be read together and the lower signs together in each formula:

$$\left.\begin{aligned}
&(1)\quad |a_{\overset{\frown}{\alpha}\overset{\frown}{\beta}\overset{\smile}{\overset{\frown}{\gamma}}\ldots.}|_n^{(p)} = \sum_{\alpha,\,\beta} \pm_{\alpha} \pm_{\beta} |a_{\alpha\beta:\overset{\smile}{\overset{\frown}{\gamma}}\ldots.}|_n^{(p-2)};\\
&(2)\quad |a_{\overset{\frown}{\alpha}\breve{\beta}\overset{\smile}{\overset{\frown}{\gamma}}\ldots.}|_n^{(p)} = \sum_{\alpha,\,\beta} \pm_{\alpha} \qquad |a_{\alpha\beta:\overset{\frown}{\overset{\smile}{\gamma}}\ldots.}|_n^{(p-2)};\\
&(3)\quad |a_{\breve{\alpha}\overset{\frown}{\beta}\overset{\smile}{\overset{\frown}{\gamma}}\ldots.}|_n^{(p)} = \sum_{\alpha,\,\beta} \qquad \pm_{\beta} |a_{\alpha\beta:\overset{\frown}{\overset{\smile}{\gamma}}\ldots.}|_n^{(p-2)};\\
&(4)\quad |a_{\breve{\alpha}\breve{\beta}\overset{\smile}{\overset{\frown}{\gamma}}\ldots.}|_n^{(p)} = \sum_{\alpha,\,\beta} \qquad\qquad |a_{\alpha\beta:\overset{\smile}{\overset{\frown}{\gamma}}\ldots.}|_n^{(p-2)}.
\end{aligned}\right\}\quad (D_2)$$

The formulas for *complete decomposition*, that is, separation into 2-way components, are:

$$\left.\begin{aligned}
&(1)\quad |a_{\ldots.\overset{\frown}{\kappa}\overset{\frown}{\lambda}}|_n^{(p)} = \Sigma \pm |a_{\ldots.:\overset{\frown}{\kappa}\overset{\frown}{\lambda}}|_n^{(2)};\\
&(2)\quad |a_{\ldots.\breve{\kappa}\overset{\frown}{\lambda}}|_n^{(p)} = \Sigma \pm |a_{\ldots.:\overset{\frown}{\kappa}\overset{\frown}{\lambda}}|_n^{(2)};\\
&(3)\quad |a_{\ldots.\overset{\frown}{\kappa}\breve{\lambda}}|_n^{(p)} = \Sigma \pm \overset{+}{|}a_{\ldots.:\breve{\kappa}\breve{\lambda}}\overset{+}{|}_n^{(2)};\\
&(4)\quad |a_{\ldots.\breve{\kappa}\breve{\lambda}}|_n^{(p)} = \Sigma \pm \overset{+}{|}a_{\ldots.:\breve{\kappa}\breve{\lambda}}\overset{+}{|}_n^{(2)}.
\end{aligned}\right\}\quad (D_{p-2})$$

The "$\pm$" is the product of the signs of all the signant ranges before κ.

It will be seen that the 2-way determinants in (1) and (2) and the 2-way permanents in (3) and (4) have for their rows files of Δ of the p-th direction (index λ) *i. e.*, rows of Δ; and that we get determinants when λ is signant, and permanents when λ is nonsignant.

As we can put the indices of any determinant in any order before decomposing it, the formulas are general with respect to such order. This remark applies to some of the later formulas.

In general, if we wish to have r nonsignant indices $\alpha_1\alpha_2\ldots\alpha_r$, and s signant indices $\beta_1\beta_2\ldots\beta_s$ come before the colon, an index γ to come immediately after it (that is, to be the index of the range which is the *base* to which $\pm_{\beta_1}, \pm_{\beta_2}, \ldots, \pm_{\beta_s}$ relate), and wish t nonsignant indices $\delta_1\delta_2\ldots\delta_t$ and u signant indices $\varepsilon_1\varepsilon_2\ldots\varepsilon_u$ to follow γ, the result is:

$$\left.\begin{aligned} &\left| a_{\overset{\smile}{\alpha_1}\cdots\overset{\smile}{\alpha_r}\overset{\frown}{\beta_1}\cdots\overset{\frown}{\beta_s}\overset{\smile}{\overset{\frown}{\gamma}}\overset{\smile}{\delta_1}\cdots\overset{\smile}{\delta_t}\overset{\frown}{\epsilon_1}\cdots\overset{\frown}{\epsilon_u}} \right|_n^{(r+s+t+u+1)} \\ &\quad = \sum_{\alpha_1\ldots\alpha_r\beta_1\ldots\beta_s} \pm_{\beta_1}\pm_{\beta_2}\cdots\pm_{\beta_s} \left| a_{\alpha_1\ldots\alpha_r\beta_1\ldots\beta_s:\overset{\frown}{\gamma}^{u}\overset{\smile}{\delta_1}\cdots\overset{\smile}{\delta_t}\overset{\frown}{\epsilon_1}\cdots\overset{\frown}{\epsilon_u}} \right|_n^{(t+u+1)}, \end{aligned}\right\} \quad (D)$$

where $\frown^{u}$ is to mean $\frown$ or $\smile$ according as u is odd or even, regardless of whether γ was originally signant or nonsignant. That is, the signancy of γ on the right is to be so taken that there will be an even number of signant indices in the components.

There is an interesting resemblance between the behavior of the signs $\smile$ and $\frown$ and that of $+$ and $-$. See (D_1) and (D_2), where, symbolically, $\frown\frown=\smile$, $\frown\smile=\frown$, $\smile\frown=\frown$, $\smile\smile=\smile$, $\frown\frown\overset{\smile}{\frown}=\overset{\smile}{\frown}$, etc. And, generally, in (D), taking the superfixes of the α's, the β's and γ, we find that $\smile\ldots\smile\frown\ldots\frown\overset{\smile}{\frown}=\overset{\smile}{\frown}$ if s is even, and $=\overset{\frown}{\smile}$ if s is odd.

7. *Element-Multiplication.*—If from the elements of the matrices

$$\| a_{\alpha_1\alpha_2\ldots\alpha_p} \|_n^{(p)}, \quad \| b_{\beta_1\beta_2\ldots\beta_q} \|_n^{(q)}$$

we form a third matrix of class $p+q-1$ and order n, in whose rows the rows of the a-matrix and the rows of the b-matrix are found according to the prescription

$$c_{\alpha_1\ldots\alpha_{p-1}\beta_1\ldots\beta_{q-1}\mu} \equiv a_{\alpha_1\ldots\alpha_{p-1}\mu}\, b_{\beta_1\ldots\beta_{q-1}\mu},$$

then it is seen that any transversal of the c-matrix consists of a transversal of the a-matrix and a transversal of the b-matrix, every possible combination occurring just once.

Let A be either a determinant or the permanent of the a-matrix, and let B be a determinant or the permanent of the b-matrix. Let C be a determinant or the permanent of the c-matrix, according to the result when the signancy of

$\alpha_1 \ldots \alpha_{p-1}$ in A, and of $\beta_1 \ldots \beta_{q-1}$ in B is continued in C, and when μ is made nonsignant in C if it is signant or is nonsignant in both A and B, but otherwise is made signant in C:

$$\frown\frown=\smile;\quad \smile\smile=\smile;\quad \smile\frown=\frown;\quad \frown\smile=\frown.$$

Then the evident theorem is:

$$AB=C.$$

The theorem includes as special cases both of the theorems of Section 3.

In the case $\frown\frown=\smile$, if either A or B is of odd class, C has more than one nonsignant index. Thus determinants that are not full-sign determinants not only fit into the cases previously known, but also create a new case.

8. *File-Multiplication.*—Given any two determinants with signant rows:

$$A \equiv \left| a_{\breve{\alpha}_1 \ldots \breve{\alpha}_f \overset{\frown}{\alpha}_{f+1} \ldots \overset{\frown}{\alpha}_p} \right|_n^{(p)}, \qquad B \equiv \left| b_{\breve{\beta}_1 \ldots \breve{\beta}_g \overset{\frown}{\beta}_{g+1} \ldots \overset{\frown}{\beta}_q} \right|_n^{(q)},$$

let us compound every row of A into every row of B in the way familiar in the case of 2-way determinants and used in Section 4, so as to form a determinant C of class $p+q-2$ and order n, according to the prescription

$$c_{\breve{\alpha}_1 \ldots \breve{\alpha}_f \overset{\frown}{\alpha}_{f+1} \ldots \overset{\frown}{\alpha}_{p-1} \breve{\beta}_1 \ldots \breve{\beta}_g \overset{\frown}{\beta}_{g+1} \ldots \overset{\frown}{\beta}_{q-1}} \equiv \sum_{\mu=1}^{n} a_{\alpha_1 \ldots \alpha_{p-1}\mu}\, b_{\beta_1 \ldots \beta_{q-1}\mu};$$

that is, combine the locants of the rows of A and B to form the locant of an element of C, and continue the signancy of the indices in those locants. Then

$$AB=C.$$

PROOF: Completely decompose A and B into:

$$\Sigma \pm_{\alpha_{f+1}} \ldots \pm_{\alpha_{p-2}} \left| a_{\alpha_1 \ldots \alpha_{p-2} : \overset{\frown}{\alpha}_{p-1} \overset{\frown}{\alpha}_p} \right|_n^{(2)},$$

$$\Sigma \pm_{\beta_{g+1}} \ldots \pm_{\beta_{q-2}} \left| b_{\beta_1 \ldots \beta_{q-2} : \overset{\frown}{\beta}_{q-1} \overset{\frown}{\beta}_q} \right|_n^{(2)};$$

$$\text{or,}\quad \Sigma \pm_\alpha \left| \begin{matrix} a_{\alpha'_1 \ldots \alpha'_{p-2}11} & a_{\alpha'_1 \ldots \alpha'_{p-2}12} & \cdots \\ a_{\alpha''_1 \ldots \alpha''_{p-2}21} & a_{\alpha''_1 \ldots \alpha''_{p-2}22} & \cdots \\ \cdots & \cdots & \cdots \end{matrix} \right|_n, \quad \Sigma \pm_\beta \left| \begin{matrix} b_{\beta'_1 \ldots \beta'_{q-2}11} & b_{\beta'_1 \ldots \beta'_{q-2}12} & \cdots \\ b_{\beta''_1 \ldots \beta''_{q-2}21} & b_{\beta''_1 \ldots \beta''_{q-2}22} & \cdots \\ \cdots & \cdots & \cdots \end{matrix} \right|_n.$$

Multiply rowwise:

$$AB = \Sigma \pm_\alpha \pm_\beta \left| \begin{matrix} \sum\limits_\mu a_{\alpha'_1 \ldots \alpha'_{p-2}1\mu}\, b_{\beta'_1 \ldots \beta'_{q-2}1\mu} & \sum\limits_\mu a_{\alpha'_1 \ldots \alpha'_{p-2}1\mu}\, b_{\beta''_1 \ldots \beta''_{q-2}2\mu} & \cdots \\ \sum\limits_\mu a_{\alpha''_1 \ldots \alpha''_{p-2}2\mu}\, b_{\beta'_1 \ldots \beta'_{q-2}1\mu} & \sum\limits_\mu a_{\alpha''_1 \ldots \alpha''_{p-2}2\mu}\, b_{\beta''_1 \ldots \beta''_{q-2}2\mu} & \cdots \\ \cdots & \cdots & \cdots \end{matrix} \right|_n$$

$$= \Sigma \pm_\alpha \pm_\beta \left| \begin{matrix} c_{\alpha'_1 \ldots \alpha'_{p-2}1\,\beta'_1 \ldots \beta'_{q-2}1} & c_{\alpha'_1 \ldots \alpha'_{p-2}1\,\beta''_1 \ldots \beta''_{q-2}2} & \cdots \\ c_{\alpha''_1 \ldots \alpha''_{p-2}2\,\beta'_1 \ldots \beta'_{q-2}1} & c_{\alpha''_1 \ldots \alpha''_{p-2}2\,\beta''_1 \ldots \beta''_{q-2}2} & \cdots \\ \cdots & \cdots & \cdots \end{matrix} \right|_n.$$

The determinants in this sum are not components of C; but the $n!$ transversals of any one of them are $n!$ of the transversals of C, and the $(n!)^{p+q-4} \cdot n!$

transversals of all are the $(n!)^{p+q-3}$ transversals of C. As to sign, a transversal

$$C\alpha'_1\ldots\ldots\alpha'_{p-2}1\beta_1^{(r_1)}\ldots\ldots\beta_{q-2}^{(r_1)}r_1 \quad C\alpha''_1\ldots\ldots\alpha''_{p-2}2\beta_1^{(r_2)}\ldots\ldots\beta_{q-2}^{(r_2)}r_2 \quad \ldots\ldots$$

has the sign $\pm_\alpha \pm_{\beta(r)} \pm_r$ in C, and the sign $\pm_\alpha \pm_\beta \pm_r$ in this sum; and since there are an even number of signant indices in $\beta_{1\ldots\ldots}\beta_{q-2}$, the signs $\pm_{\beta(r)}$ and $\pm_\beta$ are alike. Therefore $AB=C$.

EXAMPLE: $A\equiv|a_{\breve{\alpha}_1\overset{\frown}{\alpha}_2\overset{\frown}{\alpha}_3}|_2^{(3)}$, $B\equiv|b_{\breve{\beta}_1\overset{\frown}{\beta}_2\overset{\frown}{\beta}_3\overset{\frown}{\beta}_4\overset{\frown}{\beta}_5}|_2^{(5)}$;

$$c_{\breve{\alpha}_1\overset{\frown}{\alpha}_2\breve{\beta}_1\overset{\frown}{\beta}_2\overset{\frown}{\beta}_3\overset{\frown}{\beta}_4}\equiv a_{\alpha_1\alpha_2 1}\,b_{\beta_1\beta_2\beta_3\beta_4 1}+a_{\alpha_1\alpha_2 2}\,b_{\beta_1\beta_2\beta_3\beta_4 2}\,.$$

(It will be convenient here and elsewhere to insert commas in locants to bring out the structure of a matrix; they may be inserted, shifted or removed at pleasure, as they do not change the meaning of a locant.)

$$A=\begin{vmatrix} a_{1,11}a_{1,12} \\ a_{2,21}a_{2,22}\end{vmatrix}+\begin{vmatrix} a_{2,11}a_{2,12} \\ a_{1,21}a_{1,22}\end{vmatrix}, \quad B=\begin{vmatrix} b_{111,11}b_{111,12} \\ b_{222,21}b_{222,22}\end{vmatrix}-\begin{vmatrix} b_{112,11}b_{112,12} \\ b_{221,21}b_{221,22}\end{vmatrix}-\ldots\ldots;$$

$$AB=\begin{vmatrix} a_{11,1}b_{1111,1}+a_{11,2}b_{1111,2} & a_{11,1}b_{2222,1}+a_{11,2}b_{2222,2} \\ a_{22,1}b_{1111,1}+a_{22,2}b_{1111,2} & a_{22,1}b_{2222,1}+a_{22,2}b_{2222,2}\end{vmatrix}$$

$$-\begin{vmatrix} a_{11,1}b_{1121,1}+a_{11,2}b_{1121,2} & a_{11,1}b_{2212,1}+a_{11,2}b_{2212,2} \\ a_{22,1}b_{1121,1}+a_{22,2}b_{1121,2} & a_{22,1}b_{2212,1}+a_{22,2}b_{2212,2}\end{vmatrix}-\ldots\ldots$$

$$=\begin{vmatrix} c_{11,1111}c_{11,2222} \\ c_{22,1111}c_{22,2222}\end{vmatrix}-\begin{vmatrix} c_{11,1121}c_{11,2212} \\ c_{22,1121}c_{22,2212}\end{vmatrix}-\ldots\ldots=C.$$

This proof depends on the rows of A and B being signant. Apart from the proof, it is plain that one of the two indices $\alpha_p\beta_q$ can not be signant and the other nonsignant, for that would give C an odd number of signant indices. And we proceed to show that both indices can not be nonsignant; from which it follows that a full-sign determinant will not serve to express the file-product of two determinants of odd class—the restriction stated by Cayley in announcing his law of multiplication.

For, take $n=2$, to simplify the statement, In the transversal

$$c_{\alpha'_1\ldots\ldots\alpha'_{p-2}1\,\beta'_1\ldots\ldots\beta'_{q-2}1} \quad c_{\alpha''_1\ldots\ldots\alpha''_{p-2}2\,\beta''_1\ldots\ldots\beta''_{q-2}2}\,,$$

we find the monomial

$$a_{\alpha'_1\ldots\ldots\alpha'_{p-2}12} \quad b_{\beta'_1\ldots\ldots\beta'_{q-2}12} \quad a_{\alpha''_1\ldots\ldots\alpha''_{p-2}22} \quad b_{\beta''_1\ldots\ldots\beta''_{q-2}22}\,,$$

which does not consist of transversals of A and B, and we also find this monomial in the transversal

$$c_{\alpha'_1\ldots\ldots\alpha'_{p-2}1\,\beta''_1\ldots\ldots\beta''_{q-2}2} \quad c_{\alpha''_1\ldots\ldots\alpha''_{p-2}2\,\beta'_1\ldots\ldots\beta'_{q-2}1}\,;$$

and these two transversals are affected with the same sign, if α_p and β_q be assumed to be nonsignant, since there are then an even number of signant indices in $\alpha_1 \ldots . \alpha_{p-1}$ and in $\beta_1 \ldots . \beta_{q-1}$. Under this assumption, therefore, $AB \mathrel{\not\equiv} C$. The reasoning applies to every order of determinant and to every such monomial; if α_p and β_q in that monomial take one value v_1 times, and another value v_2 times, and so on, with $v_1 + v_2 + \ldots . = n$, then we shall find that monomial in $v_1 ! \, v_2 ! \ldots .$ transversals having the same sign prefixed.

It is instructive to compare the matrices resulting from element- and file-multiplication. If the polynomial elements of the latter matrix be converted into nonsignant files by deleting the $+$ signs, we have the former matrix, as is shown by the prescriptions.

9. *Crossed Decomposition.*—Let us generalize, in form and in substance, the development of a determinant which is found in the proof of the law of file-multiplication in the previous section. Beginning with the decomposition of any determinant or permanent

$$A \equiv \left| a_{\alpha_1 \ldots . \alpha_p} \right|_n^{(p)}$$

into

$$\Sigma \pm \left| a_{\alpha_1 \ldots . \alpha_{p-2} : \alpha_{p-1} \alpha_p} \right|_n^{(2)},$$

in each component alike let us cause any set of indices before the colon, provided that among them there are an even number of signant indices or else no signant indices, to take their n sets of values not in the n rows but in the n columns (as in the c-determinants in the above proof). Let us call the new determinants or permanents, with the signs of the components from which they were formed, *crossed components*. Example, $\left| a_{\breve{\alpha}_1 \hat{\alpha}_2 \hat{\alpha}_3 \hat{\alpha}_4 \hat{\alpha}_5} \right|_2^{(5)}$; a component, $\begin{vmatrix} a_{111,11} a_{111,12} \\ a_{222,21} a_{222,22} \end{vmatrix}$; the derived crossed component, $\begin{vmatrix} a_{111,11} a_{122,12} \\ a_{211,21} a_{222,22} \end{vmatrix}$.

The sum of these crossed components is equal to A. For, let the transposed indices be $\alpha_{h+1} \ldots . \alpha_{p-2}$, and consider any transversal of A:

$$a_{\alpha_1' \ldots . \alpha_{p-2}' 1 \alpha_p'} \, a_{\alpha_1'' \ldots . \alpha_{p-2}'' 2 \alpha_p''} \cdots a_{\alpha_1^{(n)} \ldots . \alpha_{p-2}^{(n)} n \alpha_p^{(n)}}.$$

Let $r_1 r_2 \ldots . r_n$ be such a permutation of $12 \ldots . n$ that

$$\alpha_p^{(r_1)} = 1, \; \alpha_p^{(r_2)} = 2, \; \ldots ., \; \alpha_p^{(r_n)} = n.$$

This transversal will be found once and only once, in and only in that crossed component whose main diagonal is

$$a_{\alpha_1' \ldots . \alpha_h' \alpha_{h+1}^{(r_1)} \ldots . \alpha_{p-2}^{(r_1)} 11} \, a_{\alpha_1'' \ldots . \alpha_h'' \alpha_{h+1}^{(r_2)} \ldots . \alpha_{p-2}^{(r_2)} 22} \cdots a_{\alpha_1^{(n)} \ldots . \alpha_h^{(n)} \alpha_{h+1}^{(r_n)} \ldots . \alpha_{p-2}^{(r_n)} nn},$$

i. e., in the crossed component

$$\pm \begin{vmatrix} a_{\alpha_1' \ldots . \alpha_h' \alpha_{h+1}^{(r_1)} \ldots . \alpha_{p-2}^{(r_1)} 11} & a_{\alpha_1' \ldots . \alpha_h' \alpha_{h+1}^{(r_2)} \ldots . \alpha_{p-2}^{(r_2)} 12} \ldots . \\ a_{\alpha_1'' \ldots . \alpha_h'' \alpha_{h+1}^{(r_1)} \ldots . \alpha_{p-2}^{(r_1)} 21} & a_{\alpha_1'' \ldots . \alpha_h'' \alpha_{h+1}^{(r_2)} \ldots . \alpha_{p-2}^{(r_2)} 22} \ldots . \\ \ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots \end{vmatrix}_n;$$

and the sign will be right, for if by the symbol

$$\pm\widehat{\alpha}^{(r_1\dots\dots r_n)}_{h+1\dots\dots p-2}$$

we denote the product of the signs of the signant ranges among $\alpha_{h+1}\dots\dots\alpha_{p-2}$ when they take their values in the order indicated by the symbols (r_1), (r_2), $\dots\dots$, (r_n), then we have the equation

$$\pm\widehat{\alpha}^{(\prime\dots\dots n)}_{1\dots\dots h}\pm\widehat{\alpha}^{(\prime\dots\dots n)}_{h+1\dots\dots p-2}\pm\alpha^{(\prime\dots\dots n)}_{p}=\pm\widehat{\alpha}^{(\prime\dots\dots n)}_{1\dots\dots h}\pm\widehat{\alpha}^{(r_1\dots\dots r_n)}_{h+1\dots\dots p-2}\pm\alpha^{(\prime\dots\dots n)}_{p}.$$

Generalizing in substance, let us decompose any determinant or permanent A into components of class q:

$$A=\Sigma\pm\left|a_{\alpha_{11}\dots\dots\alpha_{1h_1}\alpha_{21}\dots\dots\alpha_{2h_2}\dots\dots\alpha_{q1}\dots\dots\alpha_{qh_q}:\alpha_1\alpha_2\dots\dots\alpha_q}\right|^{(q)}_{n},$$

the indices before the colon being separated into q groups with an even number of signant indices (0 even) in each group other than the first. Alter each component by causing each group other than the first to take its n sets of values not in the n 1st-way layers, but in the n s-th-way layers for the s-th group. By this alteration the main diagonal is unchanged. Retain the signs of the original components for these *crossed components;* in other words, give each crossed component the sign of its main diagonal term as a term of A.

The sum of these crossed components is equal to A.

EXAMPLE: The determinant $\left|a_{\breve{\alpha}_{11}\alpha_{12}\widehat{\alpha}_{21}\widehat{\alpha}_{22}\widehat{\alpha}_{31}\widehat{\alpha}_{32}\widehat{\alpha}_1\widehat{\alpha}_2\widehat{\alpha}_3}\right|^{(9)}_{2}$ has 3-way components,

$$\Sigma\pm_{\alpha_{12}}\pm_{\alpha_{21}}\pm_{\alpha_{22}}\pm_{\alpha_{31}}\pm_{\alpha_{32}}\left|a_{\alpha_{11}\alpha_{12}\alpha_{21}\alpha_{22}\alpha_{31}\alpha_{32}}:\breve{\alpha}_1\widehat{\alpha}_2\widehat{\alpha}_3\right|^{(3)}_{2}.$$

A component: $+\left|\begin{array}{cc|cc} a_{11,11,11,,111} & a_{11,11,11,,112} & a_{22,22,22,,211} & a_{22,22,22,,212} \\ a_{11,11,11,,121} & a_{11,11,11,,122} & a_{22,22,22,,221} & a_{22,22,22,,222} \end{array}\right|$; the corresponding crossed component:

$$+\left|\begin{array}{cc|cc} a_{11,11,11,,111} & a_{11,11,22,,112} & a_{22,11,11,,211} & a_{22,11,22,,212} \\ a_{11,22,11,,121} & a_{11,22,22,,122} & a_{22,22,11,,221} & a_{22,22,22,,222} \end{array}\right|.$$

PROOF: Let $\prod\limits_{t=1}^{n} a_{\alpha^{(t)}_{11}\dots\dots\alpha^{(t)}_{1h_1}\alpha^{(t)}_{21}\dots\dots\alpha^{(t)}_{2h_2}\dots\dots\alpha^{(t)}_{q1}\dots\dots\alpha^{(t)}_{qh_q}t\alpha^{(t)}_2\dots\dots\alpha^{(t)}_q}$ be any transversal of A. Take $q-1$ permutations of $12\dots\dots n$, viz., $r_{s1}r_{s2}\dots\dots r_{sn}$, $s=2, 3, \dots\dots, q$, such that

$$\alpha^{(r_{s1})}_{s}=1,\ \alpha^{(r_{s2})}_{s}=2,\ \dots\dots,\ \alpha^{(r_{sn})}_{s}=n.$$

The transversal will be found once and only once, in and only in that crossed component whose main diagonal is:

$$\prod_{t=1}^{n} a\left(\alpha^{(t)}_{11}\dots\dots\alpha^{(t)}_{1h_1}\alpha^{(r_{2t})}_{21}\dots\dots\alpha^{(r_{2t})}_{2h_2}\dots\dots\alpha^{(r_{qt})}_{q1}\dots\dots\alpha^{(r_{qt})}_{qh_q}tt\dots\dots t\right).$$

And the sign will be right, since, for each value of s,

$$\pm\widehat{\alpha}^{(\prime\dots\dots n)}_{s1\dots\dots sh_s}=\pm\widehat{\alpha}^{(r_{s1}\dots\dots r_{sn})}_{s1\dots\dots sh_s}.$$

Thus, in the example above, the transversal

$$a_{1122111121}\, a_{2211122212}$$

must be found, if at all, in the position

$$\left| \begin{array}{c|c} & a_{22,\,11,\,22,,\,212} \\ a_{11,\,22,\,11,,\,121} & \end{array} \right|$$

in a crossed component; and there is evidently one and only one crossed component in which it is found.

The development in crossed components may be written:

$$A = \sum_s (\Pi \pm \widehat{\alpha}_{s1 \ldots . sh_s}) \left| a \left\{ \begin{array}{c|c} \alpha_{11} \ldots . \alpha_{1h_1} & \alpha_1 \\ \alpha_{21} \ldots . \alpha_{2h_2} & \alpha_2 \\ \ldots\ldots\ldots & \ldots \\ \alpha_{q1} \ldots . \alpha_{qh_q} & \alpha_q \end{array} \right\} \right|_n^{(q)}.$$

10. *Raising and Lowering the Class: The "Determinant-Permanent."*—One may arbitrarily raise the class of a given determinant or permanent by introducing one or more new indices whose values are determined by the values of one or more of the original indices, suitably adjusting their signancy. For example, using Kronecker's symbol $\delta_{i_1 \ldots . i_s} = 1$ if $i_1 = \ldots . = i_s$, otherwise $=0$, we have:

$$|a_{\widehat{\beta}\widehat{\gamma}}|_2^{(2)} = |\delta_{\alpha\beta} a_{\breve{\alpha}\widehat{\beta}\widehat{\gamma}}|_2^{(3)}; \ i.\ e., \ \begin{vmatrix} a_{11}a_{12} \\ a_{21}a_{22} \end{vmatrix} = \left| \begin{array}{cc|cc} a_{11,\,1} & a_{11,\,2} & \cdot & \cdot \\ \cdot & \cdot & a_{22,\,1} & a_{22,\,2} \end{array} \right|.$$

And in general we may introduce as many nonsignant indices as we like, each having any one-to-one correspondence that we like with one of the original indices; and may then arbitrarily render signant any two of the entire set of indices which happen to have a one-to-one correspondence with each other such that both have the same sign.

In particular, if a determinant of even class have two nonsignant indices, there may be introduced a signant index which shall everywhere take the same values as one of the nonsignant indices, the latter being also made signant, and thus being *doubled*, the result being a determinant of odd class with one nonsignant index. *I. e.*,

$$|a_{\breve{\beta}\breve{\gamma} \ldots .}|_n^{(2q)} = |\delta_{\alpha\beta} a_{\widehat{\alpha}\widehat{\beta}\breve{\gamma} \ldots .}|_n^{(2q+1)}.$$

It is by this kind of a determinant of odd class that the product of two full-sign determinants $A^{(p)}$ and $B^{(q)}$ by file-multiplication has heretofore been expressed, when p and q were odd, as a determinant C of class $p+q-1$ (not $p+q-2$); the "fixed index" (nonsignant index) of A has been doubled in C,

and the "fixed index" of B has been made the "fixed index" of C. Of course C would consist largely of zeros.*

On the other hand, let us start with a full-sign determinant A, of odd class, of a type of which C in the last paragraph is a particular case; wherein a group of indices take the same values, another group take the same values, and so on, there being r groups with an even number of indices in each, s groups with an odd number in each, and t single indices, among the latter being the nonsignant index τ_t.† Give to the elements new locants by striking out all but one of each group of indices, put them into a matrix

$$\| a'_{\rho_1 \dots \rho_r \sigma_1 \dots \sigma_s \tau_1 \dots \tau_t} \|_n^{(r+s+t)},$$

and consider the determinant

$$A' \equiv \left| a'_{\breve{\rho}_1 \dots \breve{\rho}_r \widehat{\sigma}_1 \dots \widehat{\sigma}_s \widehat{\tau}_1 \dots \widehat{\tau}_{t-1} \breve{\tau}_t} \right|_n^{(r+s+t)};$$

noting that $s+t$ is necessarily odd. Evidently,

$$A' = A.$$

Now A' is not a full-sign determinant, but from A' there could be formed a function not involving anything but full-sign determinants and permanents, viz., the function invented by Gegenbauer † and called by him a "determinant-permanent." Decompose A', using (D) of Section 6, into

$$\Sigma \pm_{\sigma_1} \dots \pm_{\sigma_s} \pm_{\tau_1} \dots \pm_{\tau_{t-1}} \left| a'_{\sigma_1 \dots \sigma_s \tau_1 \dots \tau_{t-1} : \breve{\tau}_t \breve{\rho}_1 \dots \breve{\rho}_r} \right|_n^{(r+1)}.$$

There are $(n!)^{s+t-1}$ of these permanents, and in them the $(\sigma_1 \dots \sigma_s \tau_1 \dots \tau_{t-1})$-ranges are differently written before the τ_t-range, and prefixed to each of them is the sign which is the product of the signs of these ranges. Such being the case, we can arbitrarily construct a determinant B with purely formal elements:

$$B \equiv \left| b_{\widehat{\sigma}_1 \dots \widehat{\sigma}_s \widehat{\tau}_1 \dots \widehat{\tau}_{t-1} \breve{\tau}_t} \right|_n^{(s+t)},$$

which will have the property that if for each transversal in its expansion in terms we substitute the corresponding permanent, the resulting function will be equal to A. This function is a "determinant-permanent" of class $s+t$ and *genus* $r+1$.

If A be made of *even* class by deleting τ_t, A' will be of class $r+s+t-1$, its components will be of class r, the indices after the colon will be $\breve{\rho}_1 \dots \breve{\rho}_r$, and B will become

$$\left| b_{\widehat{\sigma}_1 \dots \widehat{\sigma}_s \widehat{\tau}_1 \dots \widehat{\tau}_{t-1} \breve{\rho}_1} \right|_n^{(s+t)};$$

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* M. Lecat, "Sur la multiplication des déterminants," *Ann. Soc. Sci. de Bruxelles*, Vol. XXXVII, Part 2, p. 285.

† Abrégé, p. 32.

† L. Gegenbauer, "Einige Sätze über Determinanten höheren Ranges," *Denkschr. Akad. Wien*, Vol. LVII (1890), p. 735. Abrégé (see Note 3), p. 32.

that is, the "determinant-permanent" will be of class $s+t$ and genus r.

In either case the "determinant-permanent" is simply a decomposition of A'.

It has been said that a "determinant-permanent" is necessarily of odd class.* But the decomposition formula (D) shows us that there are two possible decompositions of a determinant $\left| a_{\overset{\frown}{a_1}\cdots\overset{\frown}{a_f}\overset{\smile}{a_{f+1}}\cdots\overset{\smile}{a_p}} \right|_n^{(p)}$ into a sum of permanents, such as to leave only signant indices before the colon, namely:

$$\Sigma \pm_{a_1}\cdots\cdots \pm_{a_f} \left| a_{a_1\cdots a_f\,:\,\overset{\smile}{a_{f+1}}\cdots\overset{\smile}{a_p}} \right|_n^{(p-f)},$$
$$\Sigma \pm_{a_1}\cdots\cdots \pm_{a_{f-1}} \left| a_{a_1\cdots a_{f-1}\,:\,\overset{\smile}{a_f}\cdots\overset{\smile}{a_p}} \right|_n^{(p-f+1)}.$$

Let A', therefore, be decomposed into

$$\Sigma \pm_{\sigma_1}\cdots\cdots \pm_{\sigma_s} \pm_{\tau_1}\cdots\cdots \pm_{\tau_{t-2}} \left| a'_{\sigma_1\cdots\sigma_s\tau_1\cdots\tau_{t-2}\,:\,\overset{\smile}{\tau_{t-1}}(\overset{\smile}{\tau_t})\overset{\smile}{\rho_1}\cdots\overset{\smile}{\rho_r}} \right|_n^{(r+1(+1))},$$

then

$$B \equiv \left| b_{\overset{\frown}{\sigma_1}\cdots\overset{\frown}{\sigma_s}\overset{\frown}{\tau_1}\cdots\overset{\frown}{\tau_{t-1}}} \right|_n^{(s+t-1)},$$

and the "determinant-permanent" is of *even* class.

EXAMPLE:

$$A \equiv \left| a_{\overset{\frown}{a_1}\overset{\frown}{\beta_2}\overset{\frown}{\beta_3}\overset{\frown}{\beta_4}\overset{\frown}{\gamma_5}\overset{\frown}{\gamma_6}\overset{\frown}{\epsilon_7}\overset{\frown}{\epsilon_8}\overset{\frown}{\epsilon_9}\overset{\frown}{\epsilon_{10}}}\,\delta_{\beta_2\beta_3\beta_4}\,\delta_{\gamma_5\gamma_6}\,\delta_{\epsilon_7\epsilon_8\epsilon_9\epsilon_{10}} \right|_2^{(10)}.$$
$$A' \equiv \left| a'_{\overset{\frown}{a}\overset{\frown}{\beta}\overset{\smile}{\gamma}\overset{\smile}{\epsilon}} \right|_2^{(4)} = \text{(i)}\,\Sigma \pm_a \pm_\beta \left| a'_{a\beta\,:\,\overset{\smile}{\gamma}\overset{\smile}{\epsilon}} \right|_2^{(2)} = \text{(ii)}\,\Sigma \pm_a \left| a'_{a\,:\,\overset{\smile}{\beta}\overset{\smile}{\gamma}\overset{\smile}{\epsilon}} \right|_2^{(3)}.$$
$$B = \text{(i)}\left| b_{\overset{\frown}{a}\overset{\frown}{\beta}\overset{\smile}{\gamma}} \right|_2^{(3)} \cdot B = \text{(ii)}\left| b_{\overset{\frown}{a}\overset{\frown}{\beta}} \right|_2^{(2)}.$$

(i) Replace $b_{111}b_{222} \quad + b_{112}b_{221} \quad - b_{121}b_{212} \quad - b_{122}b_{211}$

by
$$\overset{+}{}\!\begin{vmatrix} a'_{1111}a'_{1112} \\ a'_{2221}a'_{2222} \end{vmatrix}^{+} + \overset{+}{}\!\begin{vmatrix} a'_{1121}a'_{1122} \\ a'_{2211}a'_{2212} \end{vmatrix}^{+} - \overset{+}{}\!\begin{vmatrix} a'_{1211}a'_{1212} \\ a'_{2121}a'_{2122} \end{vmatrix}^{+} - \overset{+}{}\!\begin{vmatrix} a'_{1221}a'_{1222} \\ a'_{2111}a'_{2112} \end{vmatrix}^{+},$$

that is, by
$$\overset{+}{}\!\begin{vmatrix} a_{1,\,111,\,11,\,1111} & a_{1,\,111,\,11,\,2222} \\ a_{2,\,222,\,22,\,1111} & a_{2,\,222,\,22,\,2222} \end{vmatrix}^{+} + \cdots\cdots.$$

(ii) Replace $\qquad b_{11}b_{22} \quad - b_{12}b_{21}$

by
$$\overset{+}{}\!\begin{vmatrix} a'_{1111}a'_{1112} & a'_{2211}a'_{2212} \\ a'_{1121}a'_{1122} & a'_{2221}a'_{2222} \end{vmatrix}^{+} - \overset{+}{}\!\begin{vmatrix} a'_{2111}a'_{2112} & a'_{1211}a'_{1212} \\ a'_{2121}a'_{2122} & a'_{1221}a'_{1222} \end{vmatrix}^{+},$$

and translate into a's as in (i).

In the particular case of the determinant C described in this section, we have, first:

$$\left| a_{\overset{\smile}{a_1}\overset{\frown}{a_2}\cdots\overset{\frown}{a_p}} \right|_n^{(p)} \cdot \left| b_{\overset{\smile}{\beta_1}\overset{\frown}{\beta_2}\cdots\overset{\frown}{\beta_q}} \right|_n^{(q)} = \left| c_{\overset{\frown}{a_1}\overset{\frown}{a_1}\overset{\frown}{a_2}\cdots\overset{\frown}{a_{p-1}}\overset{\smile}{\beta_1}\overset{\frown}{\beta_2}\cdots\overset{\frown}{\beta_{q-1}}} \right|_n^{(p+q-1)}$$
$$= \Sigma \pm_{a_2}\cdots\cdots \pm_{a_{p-1}} \pm_{\beta_2}\cdots\cdots \pm_{\beta_{q-1}} \left| c'_{a_2\cdots a_{p-1}\beta_2\cdots\beta_{q-1}\,:\,\overset{\smile}{\beta_1}\overset{\smile}{a_1}} \right|_n^{(2)}, \qquad (1)$$

which gives the product of A and B in the form of a "determinant-permanent" of class $p+q-3$ and genus 2, a form previously known. We attach the prime

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* Abrégé, p. 32.

to c to give notice that one index of the group $\alpha_1\alpha_1$ has been dropped, just as a' was used above when all but one index in each group had been deleted.

And secondly we have the new form, obtained by writing after the colon any one of the indices that are before it in (1):

$$\Sigma \pm_{\alpha_2} \cdots\cdot \pm_{\alpha_{p-1}} \pm_{\beta_2} \cdots\cdot \pm_{\beta_{q-2}} \left| c'_{\alpha_2 \ldots\ldots \alpha_{p-1}\beta_2 \ldots\ldots \beta_{q-2} : \breve{\beta}_{q-1}\breve{\beta}_1\breve{\alpha}_1} \right|_n^{(3)}, \tag{2}$$

a "determinant-permanent" of even class $p+q-4$ and genus 3.

11. *A Product Determinant.*—A certain example given by Muir was generalized by Metzler in a paper "On a Determinant Each of Whose Elements is the Product of k Factors." * E. H. Moore contributed to the subject † and mentioned that a particular case of the theorem was ascribed to Kronecker. All this work was in two dimensions. Then von Sterneck extended Kronecker's result to p dimensions. ‡ We shall extend Metzler's theorem to p dimensions, including von Sterneck's generalization as a special case.

First let us deal with two sets of determinants, $A^{(1)}, A^{(2)}, \ldots\ldots, A^{(l)}$; $B^{(1)}, B^{(2)}, \ldots\ldots, B^{(k)}$:

$$A^{(\beta)} \equiv \left| a^{(\beta)}_{\breve{\alpha}_1 \ldots\ldots \breve{\alpha}_f \widehat{\alpha}_{f+1} \ldots\ldots \widehat{\alpha}_p} \right|_k^{(p)}; \quad B^{(\alpha)} \equiv \left| b^{(\alpha)}_{\breve{\beta}_1 \ldots\ldots \breve{\beta}_g \widehat{\beta}_{g+1} \ldots\ldots \widehat{\beta}_q} \right|_l^{(q)}.$$

Form a p-way determinant $\bar{A}=A^{(1)}A^{(2)}\ldots\ldots A^{(l)}$, of order kl, by placing $A^{(1)}, A^{(2)}, \ldots\ldots, A^{(l)}$ along the main diagonal, all other elements being zeros. Using the bipartite signs of order

$$11,\ 21,\ \ldots\ldots,\ k1,\quad 12,\ 22,\ \ldots\ldots,\ k2,\ \ldots\ldots,\quad 1l,\ 2l,\ \ldots\ldots,\ kl,$$

such as Moore employs, we shall have the prescription

$$\bar{a}_{(\breve{\alpha}_1\beta)(\breve{\alpha}_2\beta)\ldots\ldots(\breve{\alpha}_f\beta)(\widehat{\alpha}_{f+1}\beta)\ldots\ldots(\widehat{\alpha}_p\beta)} \equiv a^{(\beta)}_{\alpha_1 \ldots\ldots \alpha_p}.$$

In the same way, form a q-way determinant of order kl, with $B^{(1)}$, $B^{(2)}, \ldots\ldots, B^{(k)}$; then alter its form by a rearrangement of layers, placing the l layers of the first direction containing $B^{(1)}$ in the positions $11, 12, \ldots\ldots, 1l$, those containing $B^{(2)}$ in the positions $21, 22, \ldots\ldots, 2l$, and so on, and do this for each direction. The resulting determinant $\bar{B}=B^{(1)}B^{(2)}\ldots\ldots B^{(k)}$ and the prescription is:

$$\bar{b}_{(\breve{\alpha}\beta_1)\ldots\ldots(\breve{\alpha}\beta_g)(\widehat{\alpha}\beta_{g+1})\ldots\ldots(\widehat{\alpha}\beta_q)} \equiv b^{(\alpha)}_{\beta_1 \ldots\ldots \beta_q}.$$

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* *Am. Math. Monthly*, Vol. VII (1900), p. 151.

† "A Fundamental Remark Concerning Determinantal Notations with the Evaluation of an Important Determinant of Special Form," *Annals of Math.*, Vol. (2) I, p. 177.

‡ R. D. von Sterneck, "Ausdehnung eines Kronecker'schen Satzes auf Determinanten höheren Ranges," *Rend. Palermo*, Vol. XXX (1910), p. 58. Lecat points out the fact that von Sterneck's theorem does not hold when the classes are both odd; Abrégé. p. 63. The present extension is not subject to that restriction.

Multiply together $\bar{A}$ and $\bar{B}$ by rows into a determinant U of class $p+q-2$ and order kl. The elements of U which are not zeros are monomials, since a row of $\bar{A}$ that is not blank contains nonzero elements only in the β-th set of k places, while a row of $\bar{B}$ that is not blank contains only one nonzero element in each set of k places. The a-element and b-element whose product is a u-element will be the a-element in whose locant $\alpha_p=\alpha$ and the b-element in whose locant $\beta_q=\beta$; we therefore change α to α_p and β to β_q to form the resulting prescription:

$$u_{\overset{\smile}{(\alpha_1\beta_q)}\cdots\overset{\smile}{(\alpha_f\beta_q)}\overset{\frown}{(\alpha_{f+1}\beta_q)}\cdots\overset{\frown}{(\alpha_{p-1}\beta_q)}\overset{\smile}{(\alpha_p\beta_1)}\cdots\overset{\smile}{(\alpha_p\beta_g)}\overset{\frown}{(\alpha_p\beta_{g+1})}\cdots\overset{\frown}{(\alpha_p\beta_{q-1})}}\equiv a^{(\beta_q)}_{\alpha_1\ldots\alpha_p}\, b^{(\alpha_p)}_{\beta_1\ldots\beta_q},$$

where at least $(\alpha_{p-1}\beta_q)$ and $(\alpha_p\beta_{q-1})$ are signant.

This gives the theorem

$$U=A^{(1)}\ldots A^{(l)}B^{(1)}\ldots B^{(k)}.$$

For 2-way determinants the prescription becomes

$$u_{\overset{\frown}{(\alpha_1\beta_2)}\overset{\frown}{(\alpha_2\beta_1)}}\equiv a^{(\beta_2)}_{\alpha_1\alpha_2}\, b^{(\alpha_2)}_{\beta_1\beta_2},$$

which agrees with the theorem designated T_2 by Moore.

If $A^{(1)}=A^{(2)}=\ldots=A^{(l)}=A$, say, and $B^{(1)}=B^{(2)}=\ldots=B^{(k)}=B$, we have:

$$U=A^lB^k,$$

which is an extension of von Sterneck's theorem to less than full-sign determinants.

EXAMPLE: $p=3$, $q=3$, $k=2$, $l=2$. $A'\equiv|a'_{\breve{\alpha}_1\hat{\alpha}_2\hat{\alpha}_3}|^{(3)}_2$, $A''\equiv|a''_{\breve{\alpha}_1\hat{\alpha}_2\hat{\alpha}_3}|^{(3)}_2$, $B'\equiv|b'_{\breve{\beta}_1\hat{\beta}_2\hat{\beta}_3}|^{(3)}_2$, $B''\equiv|b''_{\breve{\beta}_1\hat{\beta}_2\hat{\beta}_3}|^{(3)}_2$. For $\bar{A}$ and $\bar{B}$ we have:

$$\begin{array}{c|cccc|cccc|cccc|cccc|}
 & & (11) & & & & (21) & & & & (12) & & & & (22) & & \\
 & (11) & (21) & (12) & (22) & (11) & (21) & (12) & (22) & (11) & (21) & (12) & (22) & (11) & (21) & (12) & (22) \\
\hline
(11) & a'_{111} & a'_{112} & \cdot & \cdot & a'_{211} & a'_{212} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
(21) & a'_{121} & a'_{122} & \cdot & \cdot & a'_{221} & a'_{222} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
(12) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a''_{111} & a''_{112} & \cdot & \cdot & a''_{211} & a''_{212} \\
(22) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a''_{121} & a''_{122} & \cdot & \cdot & a''_{221} & a''_{222}
\end{array};$$

$$\begin{array}{c|cccc|cccc|cccc|cccc|}
(11) & b'_{111} & \cdot & b'_{112} & \cdot & \cdot & \cdot & \cdot & \cdot & b'_{211} & \cdot & b'_{212} & \cdot & \cdot & \cdot & \cdot & \cdot \\
(21) & \cdot & \cdot & \cdot & \cdot & \cdot & b''_{111} & \cdot & b''_{112} & \cdot & \cdot & \cdot & \cdot & \cdot & b''_{211} & \cdot & b''_{212} \\
(12) & b'_{121} & \cdot & b'_{122} & \cdot & \cdot & \cdot & \cdot & \cdot & b'_{221} & \cdot & b'_{222} & \cdot & \cdot & \cdot & \cdot & \cdot \\
(22) & \cdot & \cdot & \cdot & \cdot & \cdot & b''_{121} & \cdot & b''_{122} & \cdot & \cdot & \cdot & \cdot & \cdot & b''_{221} & \cdot & b''_{222}
\end{array}.$$

The prescription is:

$$u_{\overset{\smile}{(\alpha_1\beta_3)}\overset{\frown}{(\alpha_2\beta_3)}\overset{\smile}{(\alpha_3\beta_1)}\overset{\frown}{(\alpha_3\beta_2)}}\equiv a^{(\beta_3)}_{\alpha_1\alpha_2\alpha_3}\, b^{(\alpha_3)}_{\beta_1\beta_2\beta_3}.$$

We shall condense the representation of U by writing the b-factor of each element under the a-factor. It is convenient to have the four values of the first index of u appear in the four large horizontal subdivisions; those of the second in the large vertical subdivisions; and those of the third and fourth in the horizontal and vertical lines in each square common to two intersecting subdivisions. Thus one of the sixteen rows of $\bar{A}$ is associated with each of the sixteen squares, while one of the sixteen rows of $\bar{B}$ is associated with each of the sixteen places in each square (the same place in every square):

		(11)				(21)				(12)				(22)			
		(11)	(21)	(12)	(22)	(11)	(21)	(12)	(22)	(11)	(21)	(12)	(22)	(11)	(21)	(12)	(22)
(11)	(11)	${a'_{111} \atop b'_{111}}$	·	${a'_{111} \atop b'_{121}}$	·	${a'_{121} \atop b'_{111}}$	·	${a'_{121} \atop b'_{121}}$	·	·	·	·	·	·	·	·	·
	(21)	·	${a'_{112} \atop b''_{111}}$	·	${a'_{112} \atop b''_{121}}$	·	${a'_{122} \atop b''_{111}}$	·	${a'_{122} \atop b''_{121}}$	·	·	·	·	·	·	·	·
	(12)	${a'_{111} \atop b'_{211}}$	·	${a'_{111} \atop b'_{221}}$	·	${a'_{121} \atop b'_{211}}$	·	${a'_{121} \atop b'_{221}}$	·	·	·	·	·	·	·	·	·
	(22)	·	${a'_{112} \atop b''_{211}}$	·	${a'_{112} \atop b''_{221}}$	·	${a'_{122} \atop b''_{211}}$	·	${a'_{122} \atop b''_{221}}$	·	·	·	·	·	·	·	·
(21)	(11)	${a'_{211} \atop b'_{111}}$	·	${a'_{211} \atop b'_{121}}$	·	${a'_{221} \atop b'_{111}}$	·	${a'_{221} \atop b'_{121}}$	·	·	·	·	·	·	·	·	·
	(21)	·	${a'_{212} \atop b''_{111}}$	·	${a'_{212} \atop b''_{121}}$	·	${a'_{222} \atop b''_{111}}$	·	${a'_{222} \atop b''_{121}}$	·	·	·	·	·	·	·	·
	(12)	${a'_{211} \atop b'_{211}}$	·	${a'_{211} \atop b'_{221}}$	·	${a'_{221} \atop b'_{211}}$	·	${a'_{221} \atop b'_{221}}$	·	·	·	·	·	·	·	·	·
	(22)	·	${a'_{212} \atop b''_{211}}$	·	${a'_{212} \atop b''_{221}}$	·	${a'_{222} \atop b''_{211}}$	·	${a'_{222} \atop b''_{221}}$	·	·	·	·	·	·	·	·
(12)	(11)	·	·	·	·	·	·	·	·	${a''_{111} \atop b'_{112}}$	·	${a''_{111} \atop b'_{122}}$	·	${a''_{121} \atop b'_{112}}$	·	${a''_{121} \atop b'_{122}}$	·
	(21)	·	·	·	·	·	·	·	·	·	${a''_{112} \atop b''_{112}}$	·	${a''_{112} \atop b''_{122}}$	·	${a''_{122} \atop b''_{112}}$	·	${a''_{122} \atop b''_{122}}$
	(12)	·	·	·	·	·	·	·	·	${a''_{111} \atop b'_{212}}$	·	${a''_{111} \atop b'_{222}}$	·	${a''_{121} \atop b'_{212}}$	·	${a''_{121} \atop b'_{222}}$	·
	(22)	·	·	·	·	·	·	·	·	·	${a''_{112} \atop b''_{212}}$	·	${a''_{112} \atop b''_{222}}$	·	${a''_{122} \atop b''_{212}}$	·	${a''_{122} \atop b''_{222}}$
(22)	(11)	·	·	·	·	·	·	·	·	${a''_{211} \atop b'_{112}}$	·	${a''_{211} \atop b'_{122}}$	·	${a''_{221} \atop b'_{112}}$	·	${a''_{221} \atop b'_{122}}$	·
	(21)	·	·	·	·	·	·	·	·	·	${a''_{212} \atop b''_{112}}$	·	${a''_{212} \atop b''_{122}}$	·	${a''_{222} \atop b''_{112}}$	·	${a''_{222} \atop b''_{122}}$
	(12)	·	·	·	·	·	·	·	·	${a''_{211} \atop b'_{212}}$	·	${a''_{211} \atop b'_{222}}$	·	${a''_{221} \atop b'_{212}}$	·	${a''_{221} \atop b'_{222}}$	·
	(22)	·	·	·	·	·	·	·	·	·	${a''_{212} \atop b''_{212}}$	·	${a''_{212} \atop b''_{222}}$	·	${a''_{222} \atop b''_{212}}$	·	${a''_{222} \atop b''_{222}}$

We deal next with three sets of determinants:

$$\begin{array}{l} A^{(11)}, A^{(21)}, \ldots, A^{(l1)}, A^{(12)}, \ldots, A^{(l2)}, \ldots, A^{(1m)}, \ldots, A^{(lm)}, \\ B^{(11)}, B^{(21)}, \ldots, B^{(k1)}, B^{(12)}, \ldots, B^{(k2)}, \ldots, B^{(1m)}, \ldots, B^{(km)}, \\ C^{(11)}, C^{(21)}, \ldots, C^{(k1)}, C^{(12)}, \ldots, C^{(k2)}, \ldots, C^{(1l)}, \ldots, C^{(kl)}; \end{array}$$

where

$$C^{(\alpha\beta)} \equiv \left| c^{(\alpha\beta)}_{\breve{\gamma}_1 \ldots \breve{\gamma}_h \hat{\gamma}_{h+1} \ldots \hat{\gamma}_r} \right|^{(r)}_{m}$$

Forming now $U^{(1)} = A^{(11)}A^{(21)} \ldots A^{(l1)}B^{(11)}B^{(21)} \ldots B^{(k1)}$, and so on up to $U^{(m)} = A^{(1m)} \ldots B^{(km)}$, construct $\overline{U} = U^{(1)} \ldots U^{(m)}$ (as we did $\overline{A}$) so that

$$\overline{u}_{(\breve{\alpha_1\beta_q\gamma}) \ldots (\breve{\alpha_f\beta_q\gamma})(\widehat{\alpha_{f+1}\beta_q\gamma}) \ldots (\widehat{\alpha_{p-1}\beta_q\gamma})(\breve{\alpha_p\beta_1\gamma}) \ldots (\breve{\alpha_p\beta_g\gamma})(\widehat{\alpha_p\beta_{g+1}\gamma}) \ldots (\widehat{\alpha_p\beta_{q-1}\gamma})} \equiv \bar{a}^{(\beta_q\gamma)}_{\alpha_1 \ldots \alpha_p} b^{(\alpha_p\gamma)}_{\beta_1 \ldots \beta_q};$$

and construct $\overline{C}$ (as we did $\overline{B}$) so that

$$\overline{c}_{(\breve{\alpha\beta\gamma_1}) \ldots (\breve{\alpha\beta\gamma_h})(\widehat{\alpha\beta\gamma_{h+1}}) \ldots (\widehat{\alpha\beta\gamma_r})} \equiv c^{(\alpha\beta)}_{\gamma_1 \ldots \gamma_r}.$$

Multiply together $\overline{U}$ and $\overline{C}$ by rows into a determinant V of class $p+q+r-4$ and order klm. The elements of V which are not zeros are monomials, and $\alpha_p = \alpha$, $\beta_{q-1} = \beta$, $\gamma_r = \gamma$. Detaching the suffixes of α, β and γ in the locant of an element of V, we have the prescription:

$$v \left\{ \begin{array}{llllllll} \alpha: & \breve{1} \ldots \breve{f} & \widehat{f+1} \ldots \widehat{p-1} & \breve{p} \ldots \breve{p} & \hat{p} \ldots \hat{p} & \breve{p} \ldots \breve{p} & \hat{p} \ldots \hat{p} \\ \beta: & q \ldots q & q \ldots q & 1 \ldots g & g+1 \ldots q-2 & q-1 \ldots q-1 & q-1 \ldots q-1 \\ \gamma: & r \ldots r & r \ldots r & r \ldots r & r \ldots r & 1 \ldots h & h+1 \ldots r-1 \end{array} \right\}$$

$$\equiv a^{(\beta_q\gamma_r)}_{\alpha_1 \ldots \alpha_p} b^{(\gamma_r\alpha_p)}_{\beta_1 \ldots \beta_q} c^{(\alpha_p\beta_{q-1})}_{\gamma_1 \ldots \gamma_r}.$$

Here at least the indices $(\alpha_{p-1}\beta_q\gamma_r)$ and $(\alpha_p\beta_{q-1}\gamma_{r-1})$ are signant. The superfixes on the right are the sets of two consecutive indices in the sequence

$$\beta_q \, \gamma_r \, \alpha_p \, \beta_{q-1}.$$

The prescription applicable to *four* sets of determinants is obtainable by: (i) changing v to w; (ii) subjoining to the locant the line

$$\delta: \; s \ldots s \ldots s \ldots s \; 1 \ldots i \;\; i+1 \ldots s-1,$$

the last s falling under γ_{r-2}, the 1 under γ_{r-1}, and the values p, $q-1$, and $r-1$ being continued over $\delta_2 \ldots \delta_{s-1}$; (iii) annexing on the right

$$d^{(\alpha_p\beta_{q-1}\gamma_{r-1})}_{\delta_1 \ldots \delta_s};$$

and (iv) inserting δ_s after γ_r and before α_p in the first three superfixes, so that the superfixes are now the sets of three consecutive indices in the sequence

$$\beta_q \, \gamma_r \, \delta_s \, \alpha_p \, \beta_{q-1} \, \gamma_{r-1}.$$

Each prescription is obtained in like manner from the prescription that precedes, and we have the following general theorem:

The product of r sets of determinants

$$A^{(hg_h)} \equiv \left| a^{(hg_h)}_{\smile{\alpha_{h1}}\smile{\alpha_{h2}}\ldots\smile{\alpha_{hf_h}}\frown{\alpha_{hf_h+1}}\ldots\frown{\alpha_{hp_h}}} \right|^{(p_h)}_{n_h},$$

where $h=1, 2, \ldots, r$, *and where* g_h *is the h-th set of* $r-1$ *consecutive indices in the sequence*

$$\alpha_{2p_2}\alpha_{3p_3}\ldots\alpha_{rp_r}\alpha_{1p_1}\alpha_{2,p_2-1}\alpha_{3,p_3-1}\ldots\alpha_{r-1,p_{r-1}-1},$$

is expressible as a determinant A of class $p_1+\ldots+p_r-2(r-1)$ *and order* $n_1n_2\ldots n_r$, *in which all elements are zeros excepting those given by the prescription*

$$a\left\{\begin{matrix} \overset{\smile}{(\alpha_{11}g_1)}\ldots\overset{\smile}{(\alpha_{1f_1}g_1)}\,\overset{\frown}{(\alpha_{1f_1+1}g_1)}\ldots\overset{\frown}{(\alpha_{1p_1-1}g_1)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \overset{\smile}{(\alpha_{k1}g_k)}\ldots\overset{\smile}{(\alpha_{kf_k}g_k)}\,\overset{\frown}{(\alpha_{kf_k+1}g_k)}\ldots\overset{\frown}{(\alpha_{kp_k-2}g_k)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \overset{\smile}{(\alpha_{r1}g_r)}\ldots\overset{\smile}{(\alpha_{rf_r}g_r)}\,\overset{\frown}{(\alpha_{rf_r+1}g_r)}\ldots\overset{\frown}{(\alpha_{rp_r-1}g_r)} \end{matrix}\right\} \equiv \prod_{h=1}^{r} a^{(hg_h)}_{\alpha_{h1}\ldots\alpha_{hp_h}},$$

where $k=2, 3, \ldots, r-1$.

Remark that at least the indices $(\alpha_{1p_1-1}g_1)$ and $(\alpha_{rp_r-1}g_r)$ are signant.

If all the determinants are of two dimensions, the prescription takes the form

$$a_{(\alpha_{11}\alpha_{22}\alpha_{32}\ldots\alpha_{r2})(\alpha_{12}\alpha_{21}\alpha_{31}\ldots\alpha_{r-1,1}\alpha_{r1})} \equiv \prod_{h=1}^{r} a^{(hg_h)}_{\alpha_{h1}\alpha_{h2}},$$

the sequence to which g_h applies being

$$\alpha_{22}\alpha_{32}\ldots\alpha_{r2}\alpha_{12}\alpha_{21}\alpha_{31}\ldots\alpha_{r-1,1}.$$

This agrees with the theorem of Metzler designated T_s by Moore.

12. *An Application to Transvectants.**

THEOREM: *If the* n^{p+q} *k-th transvectants of all possible pairs of binary forms taken from two sets*

$$\|f_{\alpha_1\ldots\alpha_p}\|^{(p)}_n, \quad \|\phi_{\beta_1\ldots\beta_q}\|^{(q)}_n, \quad n \geqq k+2,$$

be made the elements of a determinant

$$C \equiv \left| c_{\smile{\alpha_1}\ldots\smile{\alpha_g}\frown{\alpha_{g+1}}\ldots\frown{\alpha_p}\smile{\beta_1}\ldots\smile{\beta_h}\frown{\beta_{h+1}}\ldots\frown{\beta_q}} \right|^{(p+q)}_n, \quad \begin{cases} p-g \text{ odd}, \\ q-h \text{ odd}, \end{cases}$$

with

$$c_{\alpha_1\ldots\alpha_p\,\beta_1\ldots\beta_q} \equiv (f_{\alpha_1\ldots\alpha_r}, \phi_{\beta_1\ldots\beta_q})^k,$$

then

$$C \equiv 0.$$

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* Special cases of this theorem have been given by L. Gegenbauer (*loc. cit.*); see Abrégé, p. 94, where an important restriction on Gegenbauer's results, not mentioned by him, is brought out.

PROOF: We can form two null determinants A and B, whose elements are k-th derivatives of the f's and ϕ's respectively, with suitable numerical factors, such that $A\cdot B=C$. Letting $m_{a_1\ldots\ldots a_p}$, $\mu_{\beta_1\ldots\ldots\beta_q}$ be the degrees of $f_{a_1\ldots\ldots a_p}$, $\phi_{\beta_1\ldots\ldots\beta_q}$, respectively, set up:

$$A\equiv\left|a_{\breve{a}_1\ldots\ldots\breve{a}_g\hat{a}_{g+1}\ldots\ldots\hat{a}_p\hat{\lambda}}\right|_n^{(p+1)},\quad a_{a_1\ldots\ldots a_p\lambda}\equiv\frac{(m_{a_1\ldots\ldots a_p}-k)\,!}{m_{a_1\ldots\ldots a_p}\,!}\cdot\frac{\partial^k f_{a_1\ldots\ldots a_p}}{\partial_{x_1}^{k-\lambda+1}\,\partial_{x_2}^{\lambda-1}};$$

$$B\equiv\left|b_{\breve{\beta}_1\ldots\ldots\breve{\beta}_h\hat{\beta}_{h+1}\ldots\ldots\hat{\beta}_q\hat{\lambda}}\right|_n^{(q+1)},\quad b_{\beta_1\ldots\ldots\beta_q\lambda}\equiv(-1)^{\lambda-1}\binom{k}{\lambda-1}\frac{(\mu_{\beta_1\ldots\ldots\beta_q}-k)\,!}{\mu_{\beta_1\ldots\ldots\beta_q}\,!}\cdot\frac{\partial^k\phi_{\beta_1\ldots\ldots\beta_q}}{\partial_{x_1}^{\lambda-1}\partial_{x_2}^{k-\lambda+1}}.$$

These prescriptions fill only the first $k+1$ places in each row (file of the last direction), and we shall fill the remaining places in each row with zeros, the result being that one or more layers in A and in B will consist of zeros, whence $A=0$ and $B=0$. Obviously $A\cdot B=C$, the multiplication being of row into row.

COROLLARY 1. *If the n^4 k-th transvectants of all possible pairs of binary forms taken from two sets*

$$\left\|\begin{matrix} f_{11}\cdots\cdot f_{1n}\\ \cdots\cdots\cdots\\ f_{n1}\cdots\cdot f_{nn}\end{matrix}\right\|,\quad \left\|\begin{matrix} \phi_{11}\cdots\cdot \phi_{1n}\\ \cdots\cdots\cdots\\ \phi_{n1}\cdots\cdot \phi_{nn}\end{matrix}\right\|,\quad n\geqq k+2.$$

be made the elements of a 4-way determinant with two signant indices:

$$C\equiv\left|c_{\breve{a}_1\hat{a}_2\breve{\beta}_1\hat{\beta}_2}\right|_n^{(4)},\quad c_{a_1a_2\beta_1\beta_2}\equiv(f_{a_1a_2},\,\phi_{\beta_1\beta_2})^k,$$

then $$C\equiv 0.$$

COROLLARY 2. *If the n^2 k-th transvectants of all possible pairs of binary forms taken from two sets*

$$f_1, f_2, \ldots\ldots, f_n,\quad \phi_1, \phi_2, \ldots\ldots, \phi_n,\quad n\geqq k+2,$$

be made the elements of a 2-way determinant

$$C\equiv\left|\begin{matrix}(f_1,\,\phi_1)^k\cdots\cdot(f_1,\,\phi_n)^k\\ \cdots\cdots\cdots\cdots\cdots\\ (f_n,\,\phi_1)^k\cdots\cdot(f_n,\,\phi_n)^k\end{matrix}\right|_n,$$

then $$C\equiv 0.$$

This corollary includes as a special case Gordan's result:

$$\left|\begin{matrix}(f_1,\,\phi_1)^2\cdots\cdot(f_1,\,\phi_4)^2\\ \cdots\cdots\cdots\cdots\cdots\\ (f_4,\,\phi_1)^2\cdots\cdot(f_4,\,\phi_4)^2\end{matrix}\right|_4\equiv 0.$$

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